1/ Math 112 : Introductory Real Analysis

§ Sequences and series of functions

Def Suppose $\{f_n\}$, $n=1,2,3,\cdots$ is a sequence of functions defined on a set E, and suppose that the sequence of numbers $\{f_n(x)\}$ converges for every $x \in E$.

Then we can define a function f by

$$f(x) = \lim_{n\to\infty} f_n(x) \qquad (x \in E).$$

We say that $\{f_n\}$ converges to f pointwise. Series of functions are defined similarly.

Q. What kind of properties of the functions f_n (continuity, differentiability, integrability, etc.) are preserved under the limit ?

Ex Let $f_n(x) = \dfrac{x^2}{(1+x^2)^n}$, $n = 0, 1, 2, \cdots$

and consider $f(x) := \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \dfrac{x^2}{(1+x^2)^n}$

It is clear that $f(0)=0$. For $x \neq 0$, this is a geometric series, and

$$f(x) = \begin{cases} 0 & (x=0) \\ 1+x^2 & (x\neq 0) \end{cases},$$

so a convergent series of continuous functions may not have a continuous sum.

This suggests pointwise convergence is not a good notion of convergence of functions.

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Def We say a sequence of functions $\{f_n\}$ converges uniformly on E to a function f if for every $\varepsilon > 0$ there is an integer N such that

$n \geq N$ implies $|f_n(x) - f(x)| \leq \varepsilon$ for all $x \in E$.

Rmk For a real function f on E, consider the following space of functions:

$$X := \left\{ g : E \to \mathbb{R} \;\middle|\; \sup_{x \in E} |f(x) - g(x)| < \infty \right\}.$$

Then, $d(g,h) := \sup_{x \in E} |g(x) - h(x)|$ is a metric on X.

Uniform convergence is nothing but the convergence in the metric space (X, d).

Thm The sequence of functions $\{f_n\}$ defined on E converges uniformly on E if and only if for every $\varepsilon > 0$ there exists an integer N such that

$n, m \geq N$, $x \in E$ implies $|f_n(x) - f_m(x)| < \varepsilon$.

proof) (This is just the Cauchy criterion)

($\Rightarrow$) Suppose $\{f_n\}$ converges uniformly to f. Then there is an integer N such that

$n \geq N$, $x \in E$ implies $|f_n(x) - f(x)| \leq \frac{\varepsilon}{2}$

so that

$n, m \geq N$, $x \in E$ implies $|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)| < \varepsilon$.

($\Leftarrow$) If the Cauchy condition holds, $\{f_n\}$ converges pointwise to some f.

By sending $m \to \infty$, we see that $n \geq N$, $x \in E$ implies $|f_n(x) - f(x)| \leq \varepsilon$. ■

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Thm Suppose $\lim_{n\to\infty} f_n(x) = f(x) \quad (x \in E)$.

Put $M_n = \sup_{x\in E} |f_n(x) - f(x)|$.

Then $f_n \to f$ uniformly if and only if $M_n \to 0$ as $n\to\infty$.

proof) Immediate from the definition. ■

Thm Suppose $\{f_n\}$ is a sequence of functions on E, and suppose

$$|f_n(x)| \leq M_n \quad (x\in E,\ n=1,2,3,\cdots).$$

If $\sum M_n$ converges, then $\sum f_n$ converges uniformly.

proof) This is immediate from the Cauchy criterion. ■

- Uniform convergence and continuity

Thm Suppose $f_n \to f$ uniformly on $E \subseteq X$ where X is a metric space.
Let x be a limit point of E, and suppose that

$$\lim_{t\to x} f_n(t) = A_n \quad (n=1,2,3,\cdots).$$

Then $\{A_n\}$ converges, and $\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n$.

In other words, $\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t)$.

Cor If $\{f_n\}$ is a sequence of continuous functions on E, and if $f_n \to f$ uniformly,

then f is continuous on E.

4/ proof of the theorem)

Let $\varepsilon > 0$ be given. There exists N such that

$n, m \geq N$, $t \in E$ imply $|f_n(t) - f_m(t)| < \varepsilon$.

Letting $t \to x$, we obtain $|A_n - A_m| \leq \varepsilon$ for $n, m \geq M$,

so that $\{A_n\}$ is a Cauchy sequence and therefore converges, say to A.

Next,

$$|f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A|.$$

Choose n such that $|f(t) - f_n(t)| < \frac{\varepsilon}{3}$ for all $t \in E$,

and such that $|A_n - A| \leq \frac{\varepsilon}{3}$.

Then, for this n, choose a neighborhood V of x such that

$$|f_n(t) - A_n| < \frac{\varepsilon}{3} \quad \text{if } t \in V, t \neq x.$$

It follows that $|f(t) - A| < \varepsilon$ for all $t \in V$, $t \neq x$,

and hence $\lim_{t \to x} f(t) = A$. ■